

A Note on Generalized Intuitionistic Fuzzy ψ Normed Linear Space

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Abstract

In respect of the definition of intuitionistic fuzzy n -norm [12] , the definition of generalised intuitionistic fuzzy ψ norm (in short $GIF\psi N$) is introduced over a linear space and there after a few results on generalized intuitionistic fuzzy ψ normed linear space and finite dimensional generalized intuitionistic fuzzy ψ normed linear space have been developed. Lastly, we have introduced the definitions of generalised intuitionistic fuzzy ψ continuity , sequentially intuitionistic fuzzy ψ continuity and it is proved that they are equivalent.

Keywords: Generalized intuitionistic fuzzy normed linear space, Intuitionistic fuzzy continuity, Cauchy sequence, Sequentially Intuitionistic fuzzy continuity.

2010 Mathematics Subject Classification: 03F55, 46S40.

1 Introduction

In 1965, Zadeh[14] first introduced the concept of Fuzzy set theory and thereafter it has been developed by several authors through the contribution of the different articles on this concept and applied on different branches of pure and applied mathematics. The concept of fuzzy norm was introduced by Katsaras [10] in 1984 and in 1992, Felbin[8] introduced the idea of fuzzy norm on a linear

space. Cheng-Moderson [5] introduced another idea of fuzzy norm on a linear space whose associated metric is same as the associated metric of Kramosil-Michalek [13]. Latter on Bag and Samanta [3] modified the definition of fuzzy norm of Cheng-Moderson [5] and established the concept of continuity and boundedness of a function with respect to their fuzzy norm in [4].

The authors T. Bag and S. K. Samanta [3] introduced the definition of fuzzy norm over a linear space following the definition S. C. Cheng and J. N. Mooredson [5] and they have studied finite dimensional fuzzy normed linear spaces. Also the definition of intuitionistic fuzzy n -normed linear space was introduced in the paper [12] and established a sufficient condition for an intuitionistic fuzzy n -normed linear space to be complete. In this paper, following the definition of intuitionistic fuzzy n -norm [12], the definition of generalized intuitionistic fuzzy ψ norm (in short GIF ψ N) is defined over a linear space. There after a sufficient condition is given for a generalized intuitionistic fuzzy ψ normed linear space to be complete and also it is proved that a finite dimensional generalized intuitionistic fuzzy ψ norm linear space is complete. In such spaces, it is established that a necessary and sufficient condition for a subset to be compact. Thereafter the definition of generalized intuitionistic fuzzy ψ continuity, strongly intuitionistic fuzzy ψ continuity and sequentially intuitionistic fuzzy ψ continuity are defined and proved that the concept of intuitionistic fuzzy ψ continuity and sequentially intuitionistic fuzzy ψ continuity are equivalent. There after it is shown that intuitionistic fuzzy continuous image of a compact set is again a compact set.

2 Preliminaries

We quote some definitions and statements of a few theorems which will be needed in the sequel.

Definition 2.1 [11]. A binary operation $*$: $[0, 1] \times [0, 1] \longrightarrow [0, 1]$ is continuous t - norm if $*$ satisfies the following conditions :

- (i) $*$ is commutative and associative ,
- (ii) $*$ is continuous ,
- (iii) $a * 1 = a \quad \forall a \in [0, 1]$,
- (iv) $a * b \leq c * d$ whenever $a \leq c$, $b \leq d$ and $a, b, c, d \in [0, 1]$.

Definition 2.2 [11]. A binary operation $\diamond : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is continuous t -conorm if \diamond satisfies the following conditions :

- (i) \diamond is commutative and associative ,
- (ii) \diamond is continuous ,
- (iii) $a \diamond 0 = a \quad \forall a \in [0, 1]$,
- (iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c, b \leq d$ and $a, b, c, d \in [0, 1]$.

Corollary 2.3 [12]. (a) For any $r_1, r_2 \in (0, 1)$ with $r_1 > r_2$, there exist $r_3, r_4 \in (0, 1)$ such that $r_1 * r_3 > r_2$ and $r_1 > r_4 \diamond r_2$.
 (b) For any $r_5 \in (0, 1)$, there exist $r_6, r_7 \in (0, 1)$ such that $r_6 * r_6 \geq r_5$ and $r_7 \diamond r_7 \leq r_5$.

Definition 2.4 [15] By an operation \circ on \mathbf{R}^+ we mean a two place function $\circ : [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ which is associative, commutative, non decreasing in each place and such that $a \circ 0 = a \quad \forall a \in [0, \infty)$. The most used operations on \mathbf{R}^+ are $\circ_a(s, t) = s + t$, $\circ_m(s, t) = \max\{s, t\}$, $\circ_n(s, t) = (s^n + t^n)^{\frac{1}{n}}$.

Definition 2.5 Let ψ be a function defined on the real field \mathbf{R} into itself satisfying the following properties :

- (i) $\psi(-t) = \psi(t)$ for all $t \in \mathbf{R}$
- (ii) $\psi(1) = 1$
- (iii) ψ is strictly increasing and continuous on $(0, \infty)$
- (iv) $\lim_{\alpha \rightarrow 0} \psi(\alpha) = 0$ and $\lim_{\alpha \rightarrow \infty} \psi(\alpha) = \infty$

Example 2.6 As example of such functions, consider $\psi(\alpha) = |\alpha|$;
 $\psi(\alpha) = |\alpha|^p$, $p \in \mathbf{R}^+$; $\psi(\alpha) = \frac{2\alpha^{2n}}{|\alpha|+1}$, $n \in \mathbf{N}^+$. The function ψ allows us to generalize fuzzy metric and normed space.

Definition 2.7 [9]. Let $*$ be a continuous t -norm , \diamond be a continuous t -conorm and V be a linear space over the field $F(= \mathbf{R} \text{ or } \mathbf{C})$. An **intuitionistic fuzzy norm** on V is an object of the form $A = \{ ((x, t), \mu(x, t), \nu(x, t)) : (x, t) \in V \times \mathbf{R}^+ \}$, where μ, ν are fuzzy sets on $V \times \mathbf{R}^+$, μ denotes the degree of membership and ν denotes the degree of non - membership $(x, t) \in V \times \mathbf{R}^+$ satisfying the following conditions :

- (i) $\mu(x, t) + \nu(x, t) \leq 1 \quad \forall (x, t) \in V \times \mathbf{R}^+$;

- (ii) $\mu(x, t) > 0$;
- (iii) $\mu(x, t) = 1$ if and only if $x = \theta$, θ is null vector ;
- (iv) $\mu(cx, t) = \mu(x, \frac{t}{|c|}) \quad \forall c \in F \text{ and } c \neq 0$;
- (v) $\mu(x, s) * \mu(y, t) \leq \mu(x + y, s + t)$;
- (vi) $\mu(x, \cdot)$ is non-decreasing function of \mathbf{R}^+ and $\lim_{t \rightarrow \infty} \mu(x, t) = 1$;
- (vii) $\nu(x, t) < 1$;
- (viii) $\nu(x, t) = 0$ if and only if $x = \theta$;
- (ix) $\nu(cx, t) = \nu(x, \frac{t}{|c|}) \quad \forall c \in F \text{ and } c \neq 0$;
- (x) $\nu(x, s) \diamond \nu(y, t) \geq \nu(x + y, s + t)$;
- (xi) $\nu(x, \cdot)$ is non-increasing function of \mathbf{R}^+ and $\lim_{t \rightarrow \infty} \nu(x, t) = 0$.

Definition 2.8 [9]. If A is an intuitionistic fuzzy norm on a linear space V then (V, A) is called an intuitionistic fuzzy normed linear space.

For the intuitionistic fuzzy normed linear space (V, A) , we further assume that $\mu, \nu, *, \diamond$ satisfy the following axioms :

- (xii) $\{ \frac{a}{a} \diamond \frac{a}{a} \equiv \frac{a}{a} \}$, for all $a \in [0, 1]$.
- (xiii) $\mu(x, t) > 0$, for all $t > 0 \Rightarrow x = \theta$.
- (xiv) $\nu(x, t) < 1$, for all $t > 0 \Rightarrow x = \theta$.

Definition 2.9 [9]. A sequence $\{x_n\}_n$ in an intuitionistic fuzzy normed linear space (V, A) is said to **converge** to $x \in V$ if for given $r > 0$, $t > 0$, $0 < r < 1$, there exist an integer $n_0 \in \mathbf{N}$ such that $\mu(x_n - x, t) > 1 - r$ and $\nu(x_n - x, t) < r$ for all $n \geq n_0$.

Definition 2.10 [9]. A sequence $\{x_n\}_n$ in an intuitionistic fuzzy normed linear space (V, A) is said to be **cauchy sequence** if $\lim_{n \rightarrow \infty} \mu(x_{n+p} - x_n, t) = 1$ and $\lim_{n \rightarrow \infty} \nu(x_{n+p} - x_n, t) = 0$, $p = 1, 2, 3, \dots$.

Definition 2.11 [9]. Let, (U, A) and (V, B) be two intuitionistic fuzzy normed linear space over the same field F . A mapping f from (U, A) to (V, B) is said to be **intuitionistic fuzzy continuous** at $x_0 \in U$, if for

any given $\epsilon > 0$, $\alpha \in (0, 1)$, $\exists \delta = \delta(\alpha, \epsilon) > 0$, $\beta = \beta(\alpha, \epsilon) \in (0, 1)$ such that for all $x \in U$,

$$\mu_U(x - x_0, \delta) > 1 - \beta \Rightarrow \mu_V(f(x) - f(x_0), \epsilon) > 1 - \alpha$$

$$\nu_U(x - x_0, \delta) < \beta \Rightarrow \nu_V(f(x) - f(x_0), \epsilon) < \alpha.$$

Definition 2.12 [9]. A mapping f from (U, A) to (V, B) is said to be **sequentially intuitionistic fuzzy continuous** at $x_0 \in U$, if for any sequence $\{x_n\}_n$, $x_n \in U$, $\forall n \in \mathbf{N}$ with $x_n \rightarrow x_0$ in (U, A) implies $f(x_n) \rightarrow f(x_0)$ in (V, B) , that is

$$\lim_{n \rightarrow \infty} \mu_U(x_n - x_0, t) = 1 \text{ and } \lim_{n \rightarrow \infty} \nu_U(x_n - x_0, t) = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \mu_V(f(x_n) - f(x_0), t) = 1 \text{ and } \lim_{n \rightarrow \infty} \nu_V(f(x_n) - f(x_0), t) = 0.$$

Theorem 2.13 [9]. Let f be a mapping from (U, A) to (V, B) . Then f is intuitionistic fuzzy continuous on U if and only if it is sequentially intuitionistic fuzzy continuous on U .

3 Generalized Intuitionistic Fuzzy ψ -Normed Linear space

Definition 3.1 Let $*$ be a continuous t -norm, \diamond be a continuous t -conorm and V be a linear space over the field \mathbf{R} . A **Generalized intuitionistic fuzzy ψ -norm** on V is an object of the form $A^\psi = \{((x, t), \mu(x, t), \nu(x, t)) : (x, t) \in V \times \mathbf{R}^+\}$, where μ, ν are fuzzy sets on $V \times \mathbf{R}^+$, μ denotes the degree of membership and ν denotes the degree of non-membership $(x, t) \in V \times \mathbf{R}^+$ satisfying the following conditions :

- (i) $\mu(x, t) + \nu(x, t) \leq 1 \quad \forall (x, t) \in V \times \mathbf{R}^+$;
- (ii) $\mu(x, t) > 0$;
- (iii) $\mu(x, t) = 1$ if and only if $x = \theta$, θ is null vector ;
- (iv) $\mu(\alpha x, t) = \mu(x, \frac{t}{\psi(\alpha)}) \quad \forall \alpha \in \mathbf{R} \text{ and } \alpha \neq 0$
- (v) $\mu(x, s) * \mu(y, t) \leq \mu(x + y, s \diamond t)$;

- (vi) $\mu(x, \cdot)$ is non-decreasing function of \mathbf{R}^+ and $\lim_{t \rightarrow \infty} \mu(x, t) = 1$;
- (vii) $\nu(x, t) < 1$;
- (viii) $\nu(x, t) = 0$ if and only if $x = \theta$;
- (ix) $\nu(\alpha x, t) = \nu(x, \frac{t}{\psi(\alpha)}) \quad \forall \alpha \in \mathbf{R} \text{ and } \alpha \neq 0$
- (x) $\nu(x, s) \diamond \nu(y, t) \geq \nu(x + y, s \circ t)$;
- (xi) $\nu(x, \cdot)$ is non-increasing function of \mathbf{R}^+ and $\lim_{t \rightarrow \infty} \nu(x, t) = 0$.

If A is an Generalized intuitionistic fuzzy ψ -norm on a linear space V then (V, A) is called a Generalized intuitionistic fuzzy ψ -normed linear space.

Definition 3.2 A sequence $\{x_n\}_n$ in a generalized IF ψ NLS (V, A^ψ) is said to converge to $x \in V$ if for any given $r > 0$, $t > 0$, $r \in (0, 1)$ there exists an integer $n_0 \in N$ such that $\mu(x_n - x, t) > 1 - r$ and $\nu(x_n - x, t) < r \quad \forall n \geq n_0$.

Theorem 3.3 In a Generalized intuitionistic fuzzy ψ -normed linear space (V, A^ψ) , a sequence $\{x_n\}$ converges to x if and only if $\mu(x_n - x, t) \rightarrow 1$ and $\nu(x_n - x, t) \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Fix $t > 0$. Suppose $\{x_n\}$ converges to x in (V, A^ψ) . Then for a given r , $r \in (0, 1)$, there exists an integer $n_0 \in N$ such that $\mu(x_n - x, t) > 1 - r$ and $\nu(x_n - x, t) < r$. Thus $1 - \mu(x_n - x, t) < r$ and $\nu(x_n - x, t) < r$, and hence $\mu(x_n - x, t) \rightarrow 1$ and $\nu(x_n - x, t) \rightarrow 0$ as $n \rightarrow \infty$. Conversely, if for each $t > 0$, $\mu(x_n - x, t) \rightarrow 1$ and $\nu(x_n - x, t) \rightarrow 0$ as $n \rightarrow \infty$, then for every r , $r \in (0, 1)$, there exists an integer n_0 such that $1 - \mu(x_n - x, t) < r$ and $\nu(x_n - x, t) < r \quad \forall n \geq n_0$. Thus $\mu(x_n - x, t) > 1 - r$ and $\nu(x_n - x, t) < r$ for all $n \geq n_0$. Hence $\{x_n\}$ converges to x in (V, A^ψ) .

Theorem 3.4 The limit is unique for a convergent sequence $\{x_n\}_n$ in a generalized IF ψ NLS (V, A^ψ) .

Proof. Let $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} x_n = y$. Also let $s, t \in \mathbf{R}^+$. Now,

$$\lim_{n \rightarrow \infty} x_n = x \Rightarrow \begin{cases} \lim_{n \rightarrow \infty} \mu(x_n - x, t) = 1 \\ \lim_{n \rightarrow \infty} \nu(x_n - x, t) = 0 \end{cases}$$

$$\lim_{n \rightarrow \infty} x_n = y \Rightarrow \begin{cases} \lim_{n \rightarrow \infty} \mu(x_n - y, t) = 1 \\ \lim_{n \rightarrow \infty} \nu(x_n - y, t) = 0 \end{cases}$$

$$\begin{aligned} \nu(x - y, s \circ t) &= \nu(x - x_n + x_n - y, s \circ t) \\ &\leq \nu(x - x_n, s) \diamond \nu(x_n - y, t) \\ &= \nu(-(x_n - x), s) \diamond \nu(x_n - y, t) \\ &= \nu\left(x_n - x, \frac{s}{\psi(-1)}\right) \diamond \nu(x_n - y, t) \\ &= \nu\left(x_n - x, \frac{s}{\psi(1)}\right) \diamond \nu(x_n - y, t) \\ &= \nu(x_n - x, s) \diamond \nu(x_n - y, t) \end{aligned}$$

Taking limit, we have

$$\begin{aligned} \nu(x - y, s \circ t) &\leq \lim_{n \rightarrow \infty} \nu(x_n - x, s) \diamond \lim_{n \rightarrow \infty} \nu(x_n - y, t) = 0 \\ \implies \nu(x - y, s \circ t) &= 0 \implies x - y = \underline{0} \implies x = y \end{aligned}$$

Theorem 3.5 If $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$ then $\lim_{n \rightarrow \infty} (x_n + y_n) = x + y$ in a generalized IF ψ NLS

Proof. The proof directly follows from the proof of the theorem 3 [9]

Theorem 3.6 $\lim_{n \rightarrow \infty} x_n = x$ and $k (\neq 0) \in F \implies \lim_{n \rightarrow \infty} kx_n = kx$ in a generalized IF ψ NLS (V, A^ψ)

Proof. Obvious.

Definition 3.7 A sequence $\{x_n\}_n$ in a generalized IF ψ NLS (V, A^ψ) is said to be **cauchy sequence** if for any given $r > 0$, $t > 0$, $r \in (0, 1)$ there exists an integer $n_0 \in N$ such that $\mu(x_m - x_n, t) > 1 - r$ and $\nu(x_m - x_n, t) < r \quad \forall m, n \geq n_0$.

Theorem 3.8 In a generalized IF ψ NLS (V, A^ψ) , every convergent sequence is a Cauchy sequence.

Proof. Let $\{x_n\}_n$ be a convergent sequence in the IF ψ NLS (V, A^ψ) , with $\lim_{n \rightarrow \infty} x_n = x$.
Let $(s \circ t) \in \mathbf{R}^+$ and $p = 1, 2, 3, \dots$

we have

$$\begin{aligned} \mu(x_{n+p} - x_n, s \circ t) &= \mu(x_{n+p} - x + x - x_n, s \circ t) \\ &\geq \mu(x_{n+p} - x, s) * \mu(x - x_n, t) \end{aligned}$$

$$\begin{aligned}
&= \mu(x_{n+p} - x, s) * \mu(-(x_n - x), t) \\
&= \mu(x_{n+p} - x, s) * \mu\left(x_n - x, \frac{t}{\psi(-1)}\right) \\
&= \mu(x_{n+p} - x, s) * \mu\left(x_n - x, \frac{t}{\psi(1)}\right) \\
&= \mu(x_{n+p} - x, s) * \mu(x_n - x, t)
\end{aligned}$$

Let $r > 0, t, s > 0, r \in (0, 1)$, then \exists an integer $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}
&\mu((x_{n+p} - x_n), s \circ t) \\
&\geq \mu(x_{n+p} - x, s) * \mu(x_n - x, t) \\
&= (1 - r) * (1 - r) = (1 - r) \quad \forall n \geq n_0
\end{aligned}$$

Again,

$$\begin{aligned}
\nu(x_{n+p} - x_n, s \circ t) &= \nu(x_{n+p} - x + x - x_n, s \circ t) \\
&\leq \nu(x_{n+p} - x, s) \diamond \nu(x - x_n, t) \\
&= \nu(x_{n+p} - x, s) \diamond \nu(-(x_n - x), t) \\
&= \nu(x_{n+p} - x, s) \diamond \nu\left(x_n - x, \frac{t}{\psi(-1)}\right) \\
&= \nu(x_{n+p} - x, s) \diamond \nu\left(x_n - x, \frac{t}{\psi(1)}\right) \\
&= \nu(x_{n+p} - x, s) \diamond \nu(x_n - x, t)
\end{aligned}$$

Thus, we see that

$$\begin{aligned}
&\nu((x_{n+p} - x_n), s \circ t) \\
&\leq \nu(x_{n+p} - x, s) \diamond \nu(x_n - x, t) \\
&= r \diamond r = r \quad \forall n \geq n_0
\end{aligned}$$

Thus, $\{x_n\}_n$ is a Cauchy sequence In a Generalized IF ψ NLS (V, A^ψ) .

Note 3.9 *The converse of the above theorem is not necessarily true . It can be verified by the following example .*

Example 3.10 *Let $(V, \|\cdot\|)$ be a normed linear space and define $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ for all $a, b \in (0, 1)$. For all $t > 0$, define $\mu(x, t) = \frac{t}{t + k\|x\|}$, $\nu(x, t) = \frac{k\|x\|}{t + k\|x\|}$ where $k > 0$ and $\psi(t) = |t|$. It is easy to see that $A^\psi = \{((x, t), \mu(x, t), \nu(x, t)) : (x, t) \in V \times \mathbf{R}^+\}$ is a generalized IF ψ NLS. Then*

(i) $\{x_n\}_n$ is a Cauchy sequence in $(V, \|\cdot\|)$ if and only if $\{x_n\}_n$ is a Cauchy sequence in a Generalized IF ψ NLS (V, A^ψ) .

(ii) $\{x_n\}_n$ is a convergent sequence in $(V, \|\cdot\|)$ if and only if $\{x_n\}_n$ is a convergent sequence in a Generalized IF ψ NLS (V, A^ψ) .

Proof. The verification directly follows from the example 2 of [9].

Theorem 3.11 In a generalized IF ψ NLS (V, A^ψ) , a sequence $\{x_n\}_n$ is a Cauchy sequence if and only if $\mu(x_{n+p} - x_n, t) \rightarrow 1$ and $\nu(x_{n+p} - x_n, t) \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Fix $t > 0$. Suppose $\{x_n\}$ is a Cauchy sequence in (V, A^ψ) . Then for a given $r > 0$, $r \in (0, 1)$, there exists an integer $n_0 \in N$ such that $\mu(x_{n+p} - x_n, t) > 1 - r$ and $\nu(x_{n+p} - x_n, t) < r$. Thus $1 - \mu(x_{n+p} - x_n, t) < r$ and $\nu(x_{n+p} - x_n, t) < r$, and hence $\mu(x_{n+p} - x_n, t) \rightarrow 1$ and $\nu(x_{n+p} - x_n, t) \rightarrow 0$ as $n \rightarrow \infty$.

Conversely, for each $t > 0$, suppose $\mu(x_{n+p} - x_n, t) \rightarrow 1$ and $\nu(x_{n+p} - x_n, t) \rightarrow 0$ as $n \rightarrow \infty$. Then for every $r > 0$, $r \in (0, 1)$, there exists an integer $n_0 \in N$ such that $1 - \mu(x_{n+p} - x_n, t) < r$ and $\nu(x_{n+p} - x_n, t) < r \forall n \geq n_0$. Thus $\mu(x_{n+p} - x_n, t) > 1 - r$ and $\nu(x_{n+p} - x_n, t) < r$ for all $n \geq n_0$. Hence $\{x_n\}$ is a Cauchy sequence in (V, A^ψ) .

Definition 3.12 A generalized IF ψ NLS (V, A^ψ) is said to be complete if every cauchy sequence in (V, A^ψ) is convergent.

Theorem 3.13 Let (V, A^ψ) be a generalized IF ψ NLS. A sufficient condition for the generalized IF ψ NLS (V, A^ψ) to be complete is that every Cauchy sequence in (V, A^ψ) has a convergent subsequence.

Proof. Let $\{x_n\}_n$ be a Cauchy sequence in (V, A^ψ) and $\{x_{n_k}\}_k$ be a subsequence of $\{x_n\}_n$ that converges to $x \in V$ and $s, t, s \circ t > 0$. Since $\{x_n\}_n$ is a cauchy sequence in (V, A^ψ) , We have for $r > 0$, $r \in (0, 1)$, there exists an integer $n_0 \in N$ such that

$$\mu(x_n - x_k, s) > 1 - r \text{ and } \nu(x_n - x_k, s) < r \quad \forall n, k \geq n_0$$

Again, Since $\{x_n\}_k$ converges to x , We have

$$\mu(x_{n_k} - x, t) > 1 - r \text{ and } \nu(x_{n_k} - x, t) < r \quad \forall n_k \geq n_0$$

Now,

$$\begin{aligned}
\mu(x_n - x, s \circ t) &= \mu(x_n - x_{n_k} + x_{n_k} - x, s \circ t) \\
&\geq \mu(x_n - x_{n_k}, s) * \mu(x_{n_k} - x, t) \\
&> (1 - r) * (1 - r) = (1 - r) \quad \forall n \geq n_0
\end{aligned}$$

Again, we see that

$$\begin{aligned}
\nu(x_n - x, s \circ t) &= \nu(x_n - x_{n_k} + x_{n_k} - x, s \circ t) \\
&\leq \nu(x_n - x_{n_k}, s) \diamond \nu(x_{n_k} - x, t) \\
&< r \diamond r = r \quad \forall n \geq n_0
\end{aligned}$$

Thus $\{x_n\}_n$ converges to x in (V, A^ψ) and hence (V, A^ψ) is complete.

Definition 3.14 Let (V, A^ψ) be a generalized IF ψ NLS. A subset P of V is said to be **closed** if for any sequence $\{x_n\}_n$ in P converges to $x \in P$, that is,

$$\lim_{n \rightarrow \infty} \mu(x_n - x, t) = 1 \text{ and } \lim_{n \rightarrow \infty} \nu(x_n - x, t) = 0 \implies x \in P.$$

Definition 3.15 Let (V, A^ψ) be a generalized IF ψ NLS. A subset Q of V is said to be the **closure** of P ($\subset V$) if for any $x \in Q$, there exists a sequence $\{x_n\}_n$ in P such that $\lim_{n \rightarrow \infty} \mu(x_n - x, t) = 1$ and $\lim_{n \rightarrow \infty} \nu(x_n - x, t) = 0 \quad \forall t \in \mathbf{R}^+$. We denote the set Q by \overline{P}

Definition 3.16 A subset P of a generalized IF ψ NLS is said to be **bounded** if and only if there exist $t > 0$ and $0 < r < 1$ such that $\mu(x, t) > 1 - r$ and $\nu(x, t) < r \quad \forall x \in P$.

Definition 3.17 Let (V, A^ψ) be a generalized IF ψ NLS. A subset P of V is said to be **compact** if any sequence $\{x_n\}_n$ in P has a subsequence converging to an element of P .

Let (V, A^ψ) be a generalized IF ψ NLS. We further assume that

$$(xii) \quad \left\{ \begin{array}{l} a \diamond a = a \\ a * a = a \end{array} \right\} \quad \forall a \in [0, 1]$$

$$(xiii) \quad \mu(x, t) > 0 \quad \forall t > 0 \implies x = \underline{0}$$

$$(xiv) \quad \nu(x, t) < 1 \quad \forall t > 0 \implies x = \underline{0}$$

Theorem 3.18 Let (V, A^ψ) be a generalized IF ψ NLS satisfying the condition (Xii). Every Cauchy sequence in (V, A^ψ) is bounded.

Proof. Let us consider a fixed r_0 with $0 < r_0 < 1$ and $\{x_n\}_n$ be a Cauchy sequence in a generalized IF ψ NLS (V, A^ψ) . Then $\exists n_0 \in \mathbf{N}$ such that

$$\left. \begin{array}{l} \mu(x_{n+p} - x_n, t) > 1 - r_0 \\ \nu(x_{n+p} - x_n, t) < r_0 \end{array} \right\} \quad \forall t > 0, p = 1, 2, \dots, \forall n > n_0.$$

Now we see that

$$\mu(x_{n+p} - x_n, t) > 1 - r_0 \quad \forall t > 0, p = 1, 2, \dots, \forall n > n_0$$

$$\begin{aligned} \Rightarrow \mu(-(x_n - x_{n+p}), t) &> 1 - r_0 \\ \Rightarrow \mu\left(x_n - x_{n+p}, \frac{t}{\psi(-1)}\right) &> 1 - r_0 \\ \Rightarrow \mu\left(x_n - x_{n+p}, \frac{t}{\psi(1)}\right) &> 1 - r_0 \\ \Rightarrow \mu(x_n - x_{n+p}, t) &> 1 - r_0 \end{aligned}$$

$$\Rightarrow \text{For } t' > 0 \exists n_0 = n_0(t')$$

such that $\mu(x_n - x_{n+p}, t') > 1 - r_0 \quad \forall n > n_0, p = 1, 2, \dots$

Since $\lim_{t \rightarrow \infty} \mu(x, t) = 1$, we have for each x_i , $\exists t_i > 0$ such that

$$\mu(x_i, t) > 1 - r_0 \quad \forall t \geq t_i, i = 1, 2, \dots$$

Let $t_0 = t' \circ \max\{t_1, t_2, \dots, t_{n_0}\}$. Then ,

$$\begin{aligned} \mu(x_n, t_0) &\geq \mu(x_n, t' \circ t_{n_0}) \\ &= \mu(x_n - x_{n_0} + x_{n_0}, t' \circ t_{n_0}) \\ &\geq \mu(x_n - x_{n_0}, t') * \mu(x_{n_0}, t_{n_0}) \\ &> (1 - r_0) * (1 - r_0) = 1 - r_0 \quad \forall n > n_0 \end{aligned}$$

Thus , we have

$$\mu(x_n, t_0) > 1 - r_0 \quad \forall n > n_0$$

Also , $\mu(x_n, t_0) \geq \mu(x_n, t_n) > 1 - r_0 \quad \forall n = 1, 2, \dots, n_0$

So, we have

$$\mu(x_n, t_0) > 1 - r_0 \quad \forall n = 1, 2, \dots \quad \dots \quad (1)$$

Again, we see that

$$\begin{aligned} \nu(x_{n+p} - x_n, t) &< r_0 \quad \forall t > 0, p = 1, 2, \dots \quad \forall n > n_0 \\ \Rightarrow \nu(-(x_n - x_{n+p}), t) &< r_0 \\ \Rightarrow \nu\left(x_n - x_{n+p}, \frac{t}{\psi(-1)}\right) &< r_0 \\ \Rightarrow \nu\left(x_n - x_{n+p}, \frac{t}{\psi(1)}\right) &< r_0 \\ \Rightarrow \nu(x_n - x_{n+p}, t) &< r_0 \end{aligned}$$

\implies For $t' > 0 \exists n'_0 = n'_0(t')$ such that

$$\nu(x_n - x_{n+p}, t') < r_0 \quad \forall n > n'_0, p = 1, 2, \dots$$

Since $\lim_{t \rightarrow \infty} \nu(x, t) = 0$, we have for each $x_i, \exists t'_i > 0$ such that

$$\nu(x_i, t) < r_0 \quad \forall t \geq t'_i, i = 1, 2, \dots$$

Let $t'_0 = t' \circ \max\{t'_1, t'_2, \dots, t'_{n_0}\}$. Then ,

$$\begin{aligned} \nu(x_n, t'_0) &\leq \nu(x_n, t' \circ t'_{n_0}) \\ &= \nu(x_n - x_{n'_0} + x_{n'_0}, t' \circ t'_{n_0}) \\ &\leq \nu(x_n - x_{n'_0}, t') \diamond \nu(x_{n'_0}, t'_{n_0}) \\ &< r_0 \diamond r_0 = r_0 \quad \forall n > n'_0 \end{aligned}$$

Thus , we have

$$\nu(x_n, t'_0) < r_0 \quad \forall n > n'_0$$

Also , $\nu(x_n, t'_0) \leq \nu(x_n, t'_n) < r_0 \quad \forall n = 1, 2, \dots, n'_0$

So, we have

$$\nu(x_n, t'_0) < r_0 \quad \forall n = 1, 2, \dots \quad \dots \quad (2)$$

Let $t''_0 = \max\{t_0, t'_0\}$. Hence from (1) and (2) we see that

$$\left. \begin{aligned} \mu(x_n, t''_0) &> (1 - r_0) \\ \nu(x_n, t''_0) &< r_0 \end{aligned} \right\} \quad \forall n = 1, 2, \dots$$

This implies that $\{x_n\}_n$ is bounded in (V, A^ψ)

Theorem 3.19 *In a finite dimensional generalized IF ψ NLS (V, A^ψ) satisfying the conditions (Xii), (Xiii) and (Xiv), a subset P of V is compact if and only if P is closed and bounded in (V, A) .*

Proof. \implies *part* : Proof of this part directly follows from the proof of the theorem 2.5 [3].

\Leftarrow *part* : In this part, we suppose that P is closed and bounded in the finite dimensional generalized IF ψ NLS (V, A^ψ) . To show P is compact, consider $\{x_n\}_n$, an arbitrary sequence in P . Since V is finite dimensional, let $\dim V = n$ and $\{e_1, e_2, \dots, e_n\}$ be a basis of V . So, for each $x_k, \exists \beta_1^k, \beta_2^k, \dots, \beta_n^k \in F$ such that

$$x_k = \beta_1^k e_1 + \beta_2^k e_2 + \dots + \beta_n^k e_n, k = 1, 2, \dots$$

Following the calculation of the theorem 11 [9], we can write

$x_{k_l} = \beta_1^{k_l} e_1 + \beta_2^{k_l} e_2 + \cdots + \beta_n^{k_l} e_n$ and $\beta_1 = \lim_{n \rightarrow \infty} \beta_1^{k_l}$, $\beta_2 = \lim_{n \rightarrow \infty} \beta_2^{k_l}$, \cdots , $\beta_n = \lim_{n \rightarrow \infty} \beta_n^{k_l}$ and $x = \beta_1 e_1 + \beta_2 e_2 + \cdots + \beta_n e_n$. Now suppose that for all $t > 0$, there exist $t_1, t_2, \cdots, t_k > 0$ such that $(t_1 \circ t_2 \circ \cdots \circ t_k) > 0$. Then we have

$$\begin{aligned} \mu(x_{k_l} - x, t_1 \circ t_2 \circ \cdots \circ t_k) &= \mu\left(\sum_{i=1}^n \beta_i^{k_l} e_i - \sum_{i=1}^n \beta_i e_i, t_1 \circ t_2 \circ \cdots \circ t_k\right) \\ &= \mu\left(\sum_{i=1}^n (\beta_i^{k_l} - \beta_i) e_i, t_1 \circ t_2 \circ \cdots \circ t_k\right) \\ &\geq \mu((\beta_1^{k_l} - \beta_1) e_1, t_1) * \cdots * \mu((\beta_n^{k_l} - \beta_n) e_n, t_n) \\ &= \mu\left(e_1, \frac{t_1}{\psi(\beta_1^{k_l} - \beta_1)}\right) * \cdots * \mu\left(e_n, \frac{t_n}{\psi(\beta_n^{k_l} - \beta_n)}\right) \end{aligned}$$

Since $\lim_{l \rightarrow \infty} \frac{t_i}{\psi(\beta_i^{k_l} - \beta_i)} = \infty$, we see that

$$\lim_{l \rightarrow \infty} \mu\left(e_i, \frac{t_i}{\psi(\beta_i^{k_l} - \beta_i)}\right) = 1$$

$$\begin{aligned} \Rightarrow \lim_{l \rightarrow \infty} \mu(x_{k_l} - x, t) &\geq 1 * \cdots * 1 = 1 \quad \forall t > 0 \\ \Rightarrow \lim_{l \rightarrow \infty} \mu(x_{k_l} - x, t) &= 1 \quad \forall t > 0 \quad \cdots \quad (4) \end{aligned}$$

Again, we have

$$\begin{aligned} \nu(x_{k_l} - x, t_1 \circ t_2 \circ \cdots \circ t_k) &= \nu\left(\sum_{i=1}^n \beta_i^{k_l} e_i - \sum_{i=1}^n \beta_i e_i, t_1 \circ t_2 \circ \cdots \circ t_k\right) \\ &= \nu\left(\sum_{i=1}^n (\beta_i^{k_l} - \beta_i) e_i, t_1 \circ t_2 \circ \cdots \circ t_k\right) \\ &\leq \nu((\beta_1^{k_l} - \beta_1) e_1, t_1) \diamond \cdots \diamond \nu((\beta_n^{k_l} - \beta_n) e_n, t_n) \\ &= \nu\left(e_1, \frac{t_1}{\psi(\beta_1^{k_l} - \beta_1)}\right) \diamond \cdots \diamond \nu\left(e_n, \frac{t_n}{\psi(\beta_n^{k_l} - \beta_n)}\right) \end{aligned}$$

Since $\lim_{l \rightarrow \infty} \frac{t_i}{\psi(\beta_i^{k_l} - \beta_i)} = \infty$, we see that

$$\lim_{l \rightarrow \infty} \nu\left(e_i, \frac{t_i}{\psi(\beta_i^{k_l} - \beta_i)}\right) = 0$$

$$\begin{aligned} \Rightarrow \lim_{l \rightarrow \infty} \nu(x_{k_l} - x, t) &\leq 0 \diamond \cdots \diamond 0 = 0 \quad \forall t > 0 \\ \Rightarrow \lim_{l \rightarrow \infty} \nu(x_{k_l} - x, t) &= 0 \quad \forall t > 0 \quad \cdots \quad (5) \end{aligned}$$

Thus, from (4) and (5) we see that

$$\lim_{l \rightarrow \infty} x_{k_l} = x \Rightarrow x \in A \quad [\text{Since } A \text{ is closed}].$$

$\implies A$ is compact.

4 Generalized Intuitionistic Fuzzy $\psi - \alpha$ -Normed Linear space

Theorem 4.1 Define $\|x\|_\alpha^1 = \bigwedge \{t : \mu(x, t) \geq \alpha\}$ and $\|x\|_\alpha^2 = \bigvee \{t : \nu(x, t) \leq \alpha\}$, $\alpha \in (0, 1)$. Then both $\{\|x\|_\alpha^1 : \alpha \in (0, 1)\}$ and $\{\|x\|_\alpha^2 : \alpha \in (0, 1)\}$ are ascending family of norms on V . These norms are said to be α -norm on V corresponding to the generalized IF ψ NLS A on V .

Proof. Let $\alpha \in (0, 1)$. To prove $\|x\|_\alpha^1$ is a norm on V , we will prove the followings :

$$(1) \quad \|x\|_\alpha^1 \geq 0 \quad \forall x \in V;$$

$$(2) \quad \|x\|_\alpha^1 = 0 \iff x = \underline{0};$$

$$(3) \quad \|\alpha x\|_\alpha^1 = \psi(\alpha) \|x\|_\alpha^1;$$

$$(4) \quad \|x + y\|_\alpha^1 \leq \|x\|_\alpha^1 + \|y\|_\alpha^1.$$

The proof of (1) and (2) directly follows from the proof of the theorem 2.1 [3]. So, we now prove (3) and (4).

If $\alpha = 0$ and $\psi(\alpha) = |\alpha|$ then

$$\|\alpha x\|_\alpha^1 = \|\underline{0}\|_\alpha^1 = 0 = 0 \|x\|_\alpha^1 = |\alpha| \|x\|_\alpha^1 = \psi(\alpha) \|x\|_\alpha^1$$

If $\alpha \neq 0$ then

$$\begin{aligned} \|\alpha x\|_\alpha^1 &= \bigwedge \{s : \mu(\alpha x, s) \geq \alpha\} \\ &= \bigwedge \{s : \mu\left(x, \frac{s}{\psi(\alpha)}\right) \geq \alpha\} \\ &= \bigwedge \{\psi(\alpha)s : \mu(x, s) \geq \alpha\} \\ &= \bigwedge \psi(\alpha) \{s : \mu(x, s) \geq \alpha\} \end{aligned}$$

Therefore $\|\alpha x\|_\alpha^1 = \psi(\alpha) \|x\|_\alpha^1$

$$\|x\|_\alpha^1 + \|y\|_\alpha^1$$

$$\begin{aligned}
&= \wedge \{s : \mu(x, s) \geq \alpha\} + \wedge \{t : \mu(y, t) \geq \alpha\} \\
&\geq \wedge \{s \circ t : \mu(x, s) \geq \alpha, \mu(y, t) \geq \alpha\} \\
&= \wedge \{s \circ t : \mu(x, s) * \mu(y, t) \geq \alpha * \alpha\} \\
&\geq \wedge \{s \circ t : \mu(x + y, s \circ t) \geq \alpha\} \\
&= \|x + y\|_{\alpha}^1, \text{ which proves (4).}
\end{aligned}$$

Let $0 < \alpha_1 < \alpha_2 < 1$.

$$\begin{aligned}
\|x\|_{\alpha_1}^1 &= \wedge \{t : \mu(x, t) \geq \alpha_1\} \\
\text{and } \|x\|_{\alpha_2}^1 &= \wedge \{t : \mu(x, t) \geq \alpha_2\}.
\end{aligned}$$

Since $\alpha_1 < \alpha_2$ $\{t : \mu(x, t) \geq \alpha_2\} \subset \{t : \mu(x, t) \geq \alpha_1\}$
 $\implies \wedge \{t : \mu(x, t) \geq \alpha_2\} \geq \wedge \{t : \mu(x, t) \geq \alpha_1\}$
 $\implies \|x\|_{\alpha_2}^1 \leq \|x\|_{\alpha_1}^1$. Thus, we see that $\{\|x\|_{\alpha}^1 : \alpha \in (0, 1)\}$ is an ascending family of norms on V .

Now we shall prove that $\{\|x\|_{\alpha}^2 : \alpha \in (0, 1)\}$ is also an ascending family of norms on V .

Let $\alpha \in (0, 1)$ and $x, y \in V$.

It is obvious that $\|x\|_{\alpha}^2 \leq 0$.

Let $\|x\|_{\alpha}^2 = 0$. Now,

$$\begin{aligned}
\|x\|_{\alpha}^2 = 0 &\implies \wedge \{t : \nu(x, t) \leq (1 - \alpha)\} = 0 \\
&\implies \nu(x, t) > \alpha > 0 \quad \forall t > 0 \implies x = \underline{0}.
\end{aligned}$$

Conversely, we assume that $x = \underline{0} \implies \nu(x, t) = 0 \quad \forall t > 0$
 $\implies \wedge \{t : \nu(x, t) \leq (1 - \alpha)\} = 0 \implies \|x\|_{\alpha}^2 = 0$.

If $\alpha = 0$ and $\psi(\alpha) = |\alpha|$ then

$$\|\alpha x\|_{\alpha}^1 = \|\underline{0}\|_{\alpha}^1 = 0 = 0 \|x\|_{\alpha}^1 = |\alpha| \|x\|_{\alpha}^1 = \psi(\alpha) \|x\|_{\alpha}^1$$

If $\alpha \neq 0$ then

$$\begin{aligned}
&\|\alpha x\|_{\alpha}^1 \\
&= \wedge \{s : \nu(\alpha x, s) \leq \alpha\} \\
&= \wedge \left\{s : \nu\left(x, \frac{s}{\psi(\alpha)}\right) \leq \alpha\right\} \\
&= \wedge \{\psi(\alpha)s : \nu(x, s) \leq \alpha\} \\
&= \wedge \psi(\alpha) \{s : \nu(x, s) \leq \alpha\}
\end{aligned}$$

Therefore $\|\alpha x\|_{\alpha}^1 = \psi(\alpha) \|x\|_{\alpha}^1$.

$$\begin{aligned}
&\|x\|_{\alpha}^2 + \|y\|_{\alpha}^2 \\
&= \wedge \{s : \nu(x, s) \leq \alpha\} + \wedge \{t : \nu(y, t) \leq \alpha\} \\
&\leq \wedge \{s \circ t : \nu(x, s) \leq \alpha, \nu(y, t) \leq \alpha\} \\
&= \wedge \{s \circ t : \nu(x, s) \diamond \nu(y, t) \leq \alpha \diamond \alpha\} \\
&\leq \wedge \{s \circ t : \nu(x + y, s \circ t) \leq \alpha\}
\end{aligned}$$

$$= \|x + y\|_\alpha^2,$$

$$\text{That is } \|x + y\|_\alpha^2 \leq \|x\|_\alpha^2 + \|y\|_\alpha^2 \quad \forall x, y \in V.$$

Let $0 < \alpha_1 < \alpha_2 < 1$.

Therefore, $\|x\|_{\alpha_1}^2 = \wedge \{t : \nu(x, t) \leq \alpha_1\}$ and

$\|x\|_{\alpha_2}^2 = \wedge \{t : \nu(x, t) \leq \alpha_2\}$. Since $\alpha_1 < \alpha_2$, we have

$$\{t : \nu(x, t) \leq \alpha_1\} \subset \{t : \nu(x, t) \leq \alpha_2\}$$

$$\implies \wedge \{t : \nu(x, t) \leq \alpha_1\} \leq \wedge \{t : \nu(x, t) \leq \alpha_2\}$$

$$\implies \|x\|_{\alpha_1}^2 \leq \|x\|_{\alpha_2}^2.$$

Thus we see that $\{\|x\|_\alpha^2 : \alpha \in (0, 1)\}$ is an ascending family of norms on V .

Lemma 4.2 [3] *Let (V, A^ψ) be a generalized IF ψ NLS satisfying the condition (Xiii) and $\{x_1, x_2, \dots, x_n\}$ be a finite set of linearly independent vectors of V . Then for each $\alpha \in (0, 1)$ there exists a constant $C_\alpha > 0$ such that for any scalars $\alpha_1, \alpha_2, \dots, \alpha_n$,*

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n\|_\alpha^1 \geq C_\alpha \sum_{i=1}^n |\alpha_i|$$

where $\|\cdot\|_\alpha^1$ is defined in the previous theorem.

Theorem 4.3 *Every finite dimensional in a generalized IF ψ NLS satisfying the conditions (Xii) and (Xiii) is complete.*

Proof. Let (V, A^ψ) be a finite dimensional generalized IF ψ NLS satisfying the conditions (Xii) and (Xiii). Also, let $\dim V = k$ and $\{e_1, e_2, \dots, e_k\}$ be a basis of V . Consider $\{x_n\}_n$ as an arbitrary Cauchy sequence in (V, A^ψ) .

Let $x_n = \beta_1^{(n)} e_1 + \beta_2^{(n)} e_2 + \dots + \beta_k^{(n)} e_k$ where $\beta_1^{(n)}, \beta_2^{(n)}, \dots, \beta_k^{(n)}$ are suitable scalars. Now $\exists \beta_1, \beta_2, \dots, \beta_k \in F$ (By the calculation of the theorem 2.4 [3])

Let $x = \sum_{i=1}^k \beta_i e_i$, clearly $x \in V$. Suppose that for all $t > 0$, there exist $t_1, t_2, \dots, t_k > 0$ such that $(t_1 \circ t_2 \circ \dots \circ t_k) > 0$. Then,

$$\mu(x_n - x, t_1 \circ t_2 \circ \dots \circ t_k) = \mu\left(\sum_{i=1}^k \beta_i^{(n)} e_i - \sum_{i=1}^k \beta_i e_i, t_1 \circ t_2 \circ \dots \circ t_k\right)$$

$$= \mu\left(\sum_{i=1}^k (\beta_i^{(n)} - \beta_i) e_i, t_1 \circ t_2 \circ \dots \circ t_k\right)$$

$$\begin{aligned}
&\geq \mu((\beta_1^{(n)} - \beta_1)e_1, t_1) * \cdots * \mu((\beta_k^{(n)} - \beta_k)e_k, t_k) \\
&= \mu(e_1, \frac{t_1}{\psi(\beta_1^{(n)} - \beta_1)}) * \cdots * \mu(e_k, \frac{t_k}{\psi(\beta_k^{(n)} - \beta_k)}) \\
&\text{Since } \lim_{n \rightarrow \infty} \frac{t_i}{\psi(\beta_i^{(n)} - \beta_i)} = \infty, \text{ we see that} \\
&\lim_{n \rightarrow \infty} \mu(e_i, \frac{t_i}{\psi(\beta_i^{(n)} - \beta_i)}) = 1 \\
&\implies \lim_{n \rightarrow \infty} \mu(x_n - x, t_1 \circ t_2 \circ \cdots \circ t_k) \geq 1 * \cdots * 1 = 1 \\
&\implies \lim_{n \rightarrow \infty} \mu(x_n - x, t) = 1 \quad \text{for all } t > 0.
\end{aligned}$$

Again, we have

$$\begin{aligned}
\nu(x_n - x, t_1 \circ t_2 \circ \cdots \circ t_k) &= \nu(\sum_{i=1}^k \beta_i^{(n)} e_i - \sum_{i=1}^k \beta_i e_i, \\
&\quad t_1 \circ t_2 \circ \cdots \circ t_k) \\
&= \nu(\sum_{i=1}^k (\beta_i^{(n)} - \beta_i) e_i, t_1 \circ t_2 \circ \cdots \circ t_k) \\
&\geq \nu((\beta_1^{(n)} - \beta_1)e_1, t_1) \diamond \cdots \diamond \nu((\beta_k^{(n)} - \beta_k)e_k, t_k) \\
&= \nu(e_1, \frac{t_1}{\psi(\beta_1^{(n)} - \beta_1)}) \diamond \cdots \diamond \nu(e_k, \frac{t_k}{\psi(\beta_k^{(n)} - \beta_k)}) \\
&\text{Since } \lim_{n \rightarrow \infty} \frac{t_1}{\psi(\beta_1^{(n)} - \beta_1)} = \infty, \text{ we see that} \\
&\lim_{n \rightarrow \infty} \nu(e_i, \frac{t_1}{\psi(\beta_i^{(n)} - \beta_i)}) = 0 \quad \forall i \\
&\implies \lim_{n \rightarrow \infty} \nu(x_n - x, t_1 \circ t_2 \circ \cdots \circ t_k) \leq 0 \diamond \cdots \diamond 0 = 0 \\
&\implies \lim_{n \rightarrow \infty} \nu(x_n - x, t) = 0 \quad \forall t > 0
\end{aligned}$$

Thus, we see that $\{x_n\}_n$ is an arbitrary Cauchy sequence that converges to $x \in V$ hence the generalized IF ψ NLS (V, A^ψ) is complete.

5 Generalized Intuitionistic Fuzzy

ψ -Continuous Function

Definition 5.1 Let (U, A^ψ) and (V, B^ψ) be two generalized IF ψ NLS over the same field F . A mapping f from (U, A^ψ) to (V, B^ψ) is said to be **intuitionistic fuzzy continuous** (or in short IFC) at $x_0 \in U$, if for any given $\varepsilon > 0$, $\alpha \in (0, 1)$, $\exists \delta = \delta(\alpha, \varepsilon) > 0$, $\beta = \beta(\alpha, \varepsilon) \in (0, 1)$ such that for all $x \in U$,

$$\mu_U(x - x_0, \delta) > \beta \implies \mu_V(f(x) - f(x_0), \varepsilon) > \alpha$$

and

$$\nu_U(x - x_0, \delta) < 1 - \beta \implies \nu_V(f(x) - f(x_0), \varepsilon) < 1 - \alpha.$$

If f is continuous at each point of U , f is said to be IFC on U .

Definition 5.2 A mapping f from (U, A^ψ) to (V, B^ψ) is said to be **strongly intuitionistic fuzzy continuous** (or in short **strongly IFC**) at $x_0 \in U$, if for any given $\varepsilon > 0$, $\exists \delta = \delta(\alpha, \varepsilon) > 0$ such that for all $x \in U$,

$$\begin{aligned} \mu_V(f(x) - f(x_0), \varepsilon) &\geq \mu_U(x - x_0, \delta) \quad \text{and} \\ \nu_V(f(x) - f(x_0), \varepsilon) &< \nu_U(x - x_0, \delta). \end{aligned}$$

f is said to be strongly IFC on U if f is strongly IFC at each point of U .

Definition 5.3 A mapping f from (U, A^ψ) to (V, B^ψ) is said to be **sequentially intuitionistic fuzzy continuous** (or in short **sequentially IFC**) at $x_0 \in U$, if for any sequence $\{x_n\}_n$, $x_n \in U \forall n$, with $x_n \longrightarrow x_0$ in (U, A^ψ) implies $f(x_n) \longrightarrow f(x_0)$ in (V, B^ψ) , that is, for any given $r \in (0, 1)$, $t > 0 \exists n_0 \in N$

$$\begin{aligned} \mu_U(x_n - x_0, t) &> 1 - r \text{ and } \nu_U(x_n - x_0, t) < r \forall n > n_0 \\ \implies \mu_V(f(x_n) - f(x_0), t) &> 1 - r \text{ and } \nu_V(f(x_n) - f(x_0), t) < r \forall n > n_0 \end{aligned}$$

If f is sequentially IFC at each point of U then f is said to be sequentially IFC on U .

Theorem 5.4 Let f be a mapping from (U, A^ψ) to (V, B^ψ) . If f strongly IFC then it is sequentially IFC.

Proof. Let $f : (U, A^\psi) \longrightarrow (V, B^\psi)$ be strongly IFC on U and $x_0 \in U$. Then for each $\varepsilon > 0$, $\exists \delta = \delta(x_0, \varepsilon) > 0$ such that for all $x \in U$,

$$\begin{aligned} \mu_V(f(x) - f(x_0), \varepsilon) &\geq \mu_U(x - x_0, \delta) \quad \text{and} \\ \nu_V(f(x) - f(x_0), \varepsilon) &< \nu_U(x - x_0, \delta) \end{aligned}$$

Let $\{x_n\}_n$ be a sequence in U such that $x_n \longrightarrow x_0$ in the space (U, A^ψ) , that is, for any given $r \in (0, 1)$, $t > 0 \exists n_0 \in N$ such that $\mu_U(x_n - x, t) > 1 - r$ and $\nu_V(x_n - x, t) < r \forall n > n_0$

Again, we see that

$$\begin{aligned}\mu_V(f(x_n) - f(x_0), \varepsilon) &\geq \mu_U(x_n - x_0, \delta) \quad \text{and} \\ \nu_V(f(x_n) - f(x_0), \varepsilon) &< \nu_U(x_n - x_0, \delta)\end{aligned}$$

which implies that

$$\mu_V(f(x_n) - f(x_0), \varepsilon) > 1 - r \text{ and } \nu_V(f(x_n) - f(x_0), \varepsilon) < r \quad \forall n > n_0$$

that is, $f(x_n) \rightarrow f(x_0)$ in (V, B^ψ) . This completes the proof.

Theorem 5.5 *Let f be a mapping from the generalized IF ψ NLS (U, A^ψ) to (V, B^ψ) . Then f is IFC on U if and only if it is sequentially IFC on U .*

Proof. The proof is same as the proof of the theorem 13 [9]

Theorem 5.6 *Let f be a mapping from the IFNLS (U, A^ψ) to (V, B^ψ) and D be a compact subset of U . If f IFC on U then $f(D)$ is a compact subset of V .*

Proof. Directly follows from the proof of the theorem 14 [9]

References

- [1] Atanassov, K. *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems 20 (1986) 87 - 96.
- [2] Iqbal H. Jebril and T.K. Samanta, *Fuzzy anti-normed linear space*, Journal of mathematics and Technology, 26 February, 2010.
- [3] T. Bag and S.K. Samanta, *Finite Dimensional Fuzzy Normed Linear Spaces*, The Journal of Fuzzy Mathematics Vol. 11 (2003) 687 - 705.
- [4] T. Bag and S.K. Samanta, *Fuzzy bounded linear operators*, Fuzzy Sets and Systems 151 (2005) 513 - 547.
- [5] S.C. Cheng and J.N. Mordeson, *Fuzzy Linear Operators and Fuzzy Normed Linear Spaces*, Bull. Cal. Math. Soc. 86 (1994) 429 - 436.
- [6] Bivas Dinda, T.K. Samanta and Iqbal H. Jebril, *Fuzzy Anti-norm and Fuzzy α -anti-convergence*, (Communicated)

- [7] Bivas Dinda and T.K. Samanta , *Intuitionistic Fuzzy Continuity and Uniform Convergence* , Int. J. Open Problems Compt.Math., Vol 3, No. 1 (2010) 8 - 26.
- [8] C. Felbin , *The completion of fuzzy normed linear space*, Journal of mathematical analysis and application 174(2) (1993) 428-440.
- [9] T.K. Samanta and Iqbal H. Jebril , *Finite dimentional intuitionistic fuzzy normed linear space*, Int. J. Open Problems Compt. Math., Vol 2, No. 4 (2009) 574-591.
- [10] A.K. Katsaras , *Fuzzy topological vector space*, Fuzzy Sets and Systems 12 (1984) 143 - 154.
- [11] B. Schweizer , A. Sklar, *Statistical metric space*, Pacific journal of mathematics 10 (1960) 314-334.
- [12] S. Vijayabalaji, N. Thillaigovindan, Y.B. Jun *Intuitionistic Fuzzy n-normed linear space* , Bull. Korean Math. Soc. 44 (2007) 291 - 308.
- [13] O. Kramosil, J. Michalek , *Fuzzy metric and statisticalmetric spaces*, Kybernetika 11 (1975) 326 - 334.
- [14] L.A. Zadeh *Fuzzy sets*, Information and control 8 (1965) 338-353.
- [15] N. Huang and H. Lan, *A couple of nonlinear equations with fuzzy mappings in fuzzy normed space*, Fuzzy Sets and System, 152 (2005), 209 - 222.